Proof Techniques

There are various words that are thrown around that are often not fully defined. Examples of these are words like **Negation**, **Conditional Statements**, **Converse**, **Inverse**, **Contrapositive**, and **Proof by Contradiction**. This document that you're reading will hopefully shed some light on these words.

Negation:

Think of two numbers α and β. The only restriction here is that these numbers must be real (Integers, fractions, rational and irrational numbers are fine), we don't want to deal with complex numbers just yet (Those imaginary numbers).

If I were to ask you, "Tell me what are the **distinct** possibilities of how numbers α and β relate to each other." You'd probably be wondering what, "**distinct** possibilities." even mean. Allow me to give you one answer. One distinct possibility of how the numbers α and β could be related to each other is

$$\boxed{\alpha = \beta}$$

With this, you may tell me that other, "possibilities." Are $\alpha \leq \beta$ and $\alpha \geq \beta$. However, I said **distinct.** Notice that in the possibilities of $\alpha \leq \beta$ and $\alpha \geq \beta$, these two had an, "Equal to." Portion written in the in equality? This, "Equal to." Business is already encompassed by the possibility that I've given to you, and so we'll disregard the two possibilities of $\alpha \leq \beta$ and $\alpha \geq \beta$ as these two were not distinct from the possibility that I gave you and we'd be double counting.

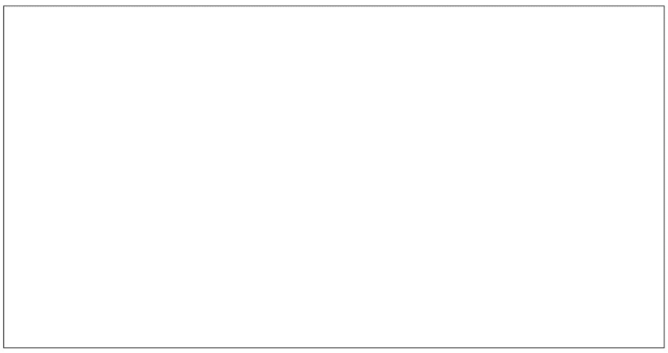
However, what if you told me $\alpha < \beta$ and $\alpha > \beta$. These two possibilities are distinct from the one that I've given you. Even better still, these two possibilities are distinct from each other. So we now have three distinct possibilities for how the two numbers of α and β relate to each other. They are

$$\boxed{\alpha = \beta}, \boxed{\alpha < \beta, \alpha > \beta}$$

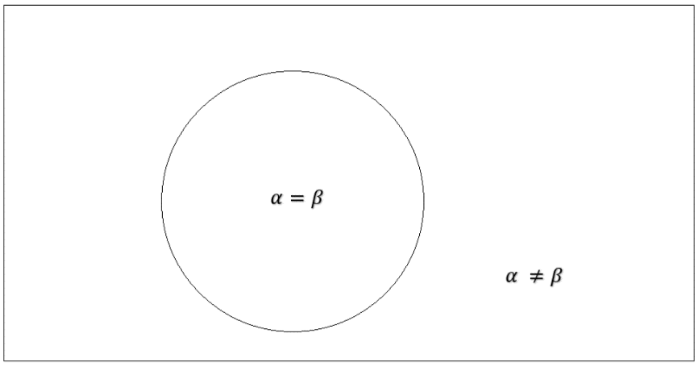
However, we can simplify this even more. Maths (Science in general) is a quest for simplicity. Saying that the numbers $\alpha < \beta$ or $\alpha > \beta$ is the same thing as saying $\alpha \neq \beta$. So we've compressed our three possibilities into two. Namely

$$\boxed{\alpha = \beta}, \boxed{\alpha \neq \beta}.$$

Let's represent this as a **Venn Diagram**. Let's assume that every possible solution of how α and β relate to each other is smeared out in this box (smeared as there are an infinite number of possibilities).



However, let's give this a little bit of structure to help us. Let's say that all of the possibilities of $\boxed{\alpha = \beta}$ is represented by a circle inside this box, and that all the possibilities of $\boxed{\alpha \neq \beta}$ is outside of the circle but still within the box. Like so



A good question now is, "How is this even helpful?" This is extremely helpful for several reasons, the main one being. If my aim is to somehow prove that I am inside the circle, I can assume I'm not in the circle and prove that that is wrong and therefore I MUST be inside the circle. On the flipside, if my aim was to prove that I am outside the circle, then I can assume that I am inside and somehow prove that that is wrong and therefore I MUST be standing outside. This idea of, "**One or the other**." Is known as **Negation** and its true power will come in to play when we explore **Proof by Contradiction**.

Conditional Statements and Contrapositive:

We've developed a brief intuition as to what Negation means. We'll now use this to develop Conditional Statements and Contrapositive. We'll start with Conditional Statements.

We've all seen it, those $P \rightarrow Q$ writings which aren't very helpful especially when the only explanation is, "If P then Q." We'd like more of an elaboration on this matter. Let's translate it to plain English. Let's assume the statement P means, "It is a bee." And that the statement Q means, "It is an insect." If we look at this in our familiar English the statement of P **implies** Q $(P \rightarrow Q)$ becomes

IF, "It is a bee." THEN, "It is an insect." This is exactly what we mean by, "Conditional Statements." The whole, "IF, THEN." Business. Notice, that, "It is an insect." (Statement Q) depends on the Condition that, "It is a bee." (Statement P). In Mathematical words, statement Q can only happen **on the condition** that statement P has happened.

I cannot write this arrow backwards, i.e. I cannot write $P \leftarrow Q$. Because IF it is an insect, THEN it is a bee is not necessarily true. An insect could be an ant, termite etc.

And this here is where **Contrapositive** comes in to play. For the Contrapositive, we'll make use of our recently learned Negation. When we say the Negation of some statement A, we usually say, "NOT A." Or simply, "$\neg A$." This symbol of $\neg$ represents the Negation.

Contrapositive is that we Negate both P and Q followed by switching the direction of the arrow. Like so

$$\neg P \leftarrow \neg Q$$

Sometimes, we even like to write it as $\neg Q \rightarrow \neg P$. How will this translate to our example of the bee?

IF it is NOT an insect ($\neg Q$), THEN it is NOT a bee ($\neg P$). Which kind of makes sense, if an animal isn't an insect, we can definitively say that it isn't a bee. It could be a fish, lizard etc.

So, the Contrapositive is, "Logically Equivalent." To the original statement.

$(Example)$:
Given that $a \in \mathbb{Z}$. ($\mathbb{Z}$ means all the integers.)
Prove that
If $a^2 - 2a + 7$ is even, then a is odd.

Solution:
We'll use Contrapositive to solve this one. As statement P is, "$a^2 - 2a + 7$ is even." And statement Q is, "a is odd." We'll Negate both these statements.

$\neg Q$: a is not odd (a is even)
$\neg P$: $a^2 - 2a + 7$ is not even (i.e, it is odd.)

$$\neg Q \rightarrow \neg P$$

As a is assumed to be an even number, that means that there is an integer k for which $a = 2k$.
This means that as $\neg P = a^2 - 2a + 7$ then $\neg P = (2k)^2 - 2(2k) + 7$.

Expanding $\neg P$, we get $\neg P = 4k^2 - 4k + 7 \equiv 2(2k^2 - 2k + 3) + 1$. This is an odd number due to the $+1$ at the end. Therefore, as the statement $\neg Q \rightarrow \neg P$ is satistfied, the statement of $P \rightarrow Q$ is also satisfied.

Converse and Inverse:

The Converse and the Inverse are what follows after Conditional statements and Contrapositive.

Original Statement	Contrapositive	Converse	Inverse
$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	$Q \rightarrow P$	$\neg P \rightarrow \neg Q$

Notice the subtle differences between Converse and Inverse.

Proof by Contradiction:

The part we've all been waiting for. We can exploit the idea of a Negation to simplify our Proofs. We'll dive right into an example to illustrate the key features of Poof by Contradiction.

$(Example)$: The Arithmetic-Geometric Mean Inequality.
If x, y are two non-negative real numbers, prove that

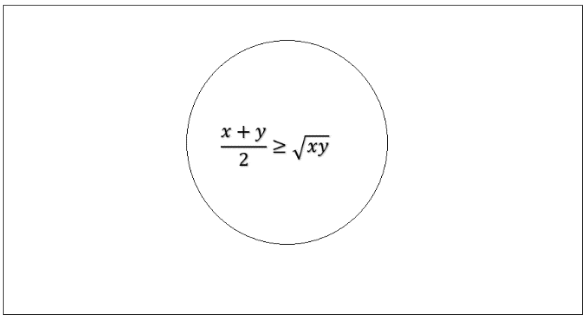
$$\frac{x+y}{2} \geq \sqrt{xy}$$

Solution:
The prime example used in Proof by Contradiction. This example is way overused as it does illustrate all the ideas we've seen thus far. Let's first decompose this into a Conditional Statement.

IF P, THEN Q. Or
IF $\underbrace{x, y\ are\ two\ non-negative\ real\ numbers}_{Statement\ P}$, THEN $\underbrace{\frac{x+y}{2} \geq \sqrt{xy}}_{Statement\ Q}$
Now that we've identified statements P and Q, let's explain the mechanics of proof by contradiction. We are currently under the assumption that the inequality holds because x and y are two non-negative real numbers. So how will we use proof by contradiction to prove this? Firstly, we need to identify the Negation.

We're currently in the $P \to Q$ form, or to put it in the pictorial (Venn Diagram) form.

Again, this rectangle represents all infinite x, y pairs. Let's partition this set into a more manageable form.

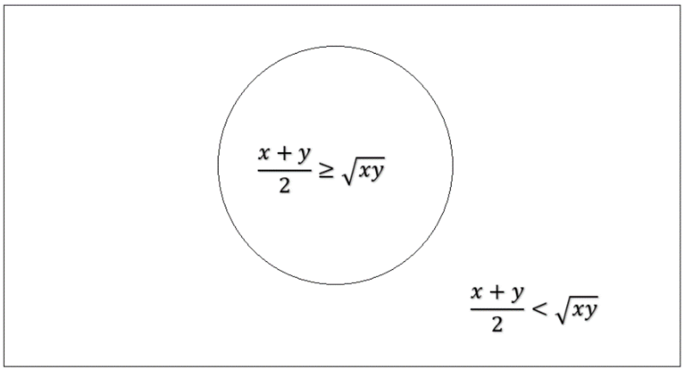


Let all the solutions of $\frac{x+y}{2} \geq \sqrt{xy}$ lie inside the circle, but how then do we prove that this is indeed the case? We'll need to use the idea of Negations to go outside of the circle.

We want to negate the inequality by, “Assuming its opposite.” What is its opposite?

$$Q: \frac{x+y}{2} \geq \sqrt{xy}$$
$$\neg Q: \frac{x+y}{2} < \sqrt{xy}$$

Notice that the inequality of the negation assumption is, “LESS THAN.” And not, “LESS THAN OR EQUAL TO.” This is because we don’t want to double count, remember we want these statements to be distinct. Our Venn Diagram then gets updated to



Hopefully you’ve noticed that we started with $P \to Q$ and by the negation assumption, we’re going to work on $P \to \neg Q$ and, “Break.” This.

This amounts to us wanting to prove that we want to be, “Inside the circle.” To do so, we’ll assume that we’re outside the circle and show that that is wrong which has to mean that we’re inside the circle.

To finish off this question, we work on the negation assumption ($\neg Q$).

$$\frac{x+y}{2} < \sqrt{xy}$$

Squaring both sides

$$\frac{(x+y)^2}{4} < xy$$

Expanding and rearranging

$$x^2 + 2xy + y^2 < 4xy$$

As x, y are non-negative, we do not worry about the inequality sign changing direction when we rearrange.

$$x^2 + 2xy - 4xy + y^2 < 0$$
$$x^2 - 2xy + y^2 < 0$$

We’re starting to see an issue here, but let’s factorise it to see the real issue.

$$(x-y)^2 < 0.$$

Now, x, y are non-negative. For any pair of numbers of the left-hand side, it will always be made positive as there is a power of 2. Even if x, y were both 0, 0 must be equal to 0 not less than 0. We’ve arrived at a contradiction. Therefore, our original statement must be true.

$(Example)$: The Diophantine equation.
In a branch of Mathematics known as Number Theory, one important family of equations are the Diophantine equations. Consider the Diophantine equation

$$x^2 - 9y^2 = 11.$$

Solutions are required in which both x and y are positive integers. By factorising or otherwise, prove by contradiction that this equation cannot have any positive integer solutions.

Solution:
Seems tough, but let us first identify statement P and statement Q.

Notice that this equation cannot be put in the $P \rightarrow Q$ form and then we work to break $P \rightarrow \neg Q$. This type of proof only has Q and $\neg Q$, much like the, "Prove $\sqrt{2}$ is irrational." Question that you must have seen countless times. In any case, we shall first assume that the solution is satisfied by positive integer solutions ($\neg Q$) and also factorise as this is a difference of perfect squares.

$$(x - 3y)(x + 3y) = 11$$

As 11 is a prime number, that means both brackets are themselves integers with one bracket being 1 and the other being 11. Let's choose the left bracket to be 1.

$$x - 3y = 1$$
$$x = 1 + 3y.$$

Putting this back into the other bracket we get

$$(x + 3y) = 11$$
$$\big((1 + 3y) + 3y\big) = 11$$
$$6y = 10$$
$$y = \frac{5}{3}.$$

This is a contradiction as both x, y must be integers and y is not an integer. Therefore the original proposition of $x^2 - 9y^2 = 11$ has no positive integer solutions must be true.